

Sharp upper bounds for the deviations from the mean of the sum of independent Rademacher random variables

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Abstract

For a fixed unit vector $a = (a_1, a_2, \dots, a_n) \in S^{n-1}$, i.e. $\sum_{i=1}^n a_i^2 = 1$, we consider the 2^n sign vectors $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \{-1, 1\}^n$ and the corresponding scalar products $a \cdot \epsilon = \sum_{i=1}^n a_i \epsilon_i$. In [1] the following old conjecture has been reformulated. It states that among the 2^n sums of the form $\sum \pm a_i$ there are not more with $|\sum_{i=1}^n \pm a_i| > 1$ than there are with $|\sum_{i=1}^n \pm a_i| \leq 1$. The result is of interest in itself, but has also an appealing reformulation in probability theory and in geometry. In this paper we will solve an extension of this problem in the uniform case where all the a 's are equal. More precisely, for S_n being a sum of n independent Rademacher random variables, we will give, for several values of ξ , precise lower bounds for the probabilities

$$P_n := \mathbb{P}\{-\xi\sqrt{n} \leq S_n \leq \xi\sqrt{n}\}$$

or equivalently for

$$Q_n := \mathbb{P}\{-\xi \leq T_n \leq \xi\},$$

where T_n is a standardized Binomial random variable with parameters n and $p = 1/2$. These lower bounds are sharp and much better than for instance the bound that can be obtained from application of the Chebishev inequality. In case $\xi = 1$ Van Zuijlen solved this problem in [3]. We remark that our bound will have nice applications in probability theory and especially in random walk theory.

1 Introduction and result

Let $\epsilon_1, \epsilon_2, \dots$, be a sequence of i.i.d. Rademacher random variables and for positive integers n let $a_n = (a_{1n}, a_{2n}, \dots, a_{nn})$ be a unit-vectors in \mathbb{R}^n , so that $\sum_{i=1}^n a_{in}^2 = 1$. The following problem has been presented in [2] and is attributed to B. Tomaszewski. In [1], Conjecture 1.1, this old problem has been reformulated as follows:

$$\mathbb{P}(|a_{1n}\epsilon_1 + a_{2n}\epsilon_2 + \dots + a_{nn}\epsilon_n| \leq 1) \geq \frac{1}{2}, \text{ for } n = 1, 2, \dots$$

This conjecture is at least 25 years old and seems still to be unsolved. In the uniform case where,

$$a_{1n} = a_{2n} = \dots = a_{nn} = n^{-1/2},$$

the maximum possible value of $\frac{S_n}{\sqrt{n}}$ is \sqrt{n} , where

$$S_n := \epsilon_1 + \epsilon_2 + \dots + \epsilon_n \tag{1}$$

and the conjecture, stating that for integers $n \geq 2$,

$$\mathbb{P}\{|S_n| \leq \sqrt{n}\} = \mathbb{P}\{|\sum_{i=1}^n \epsilon_i| \leq \sqrt{n}\} \geq 1/2,$$

has been solved recently by M.C.A. van Zuijlen. See [3]. It means that at least 50% of the probability mass is between minus one and one standard deviation from the mean, which is quite remarkable. We note that

- i) S_n can be easily expressed in terms of sums of independent Bernoulli(1/2) random variables since $(\epsilon_i + 1)/2$ are independent Bernoulli random variables and hence S_n is distributed as $2B_n - n$, where B_n is a binomial random variable with parameters n and $1/2$. It follows that S_n/\sqrt{n} is distributed as T_n , where T_n is a binomial random variable with parameters n and $p = 1/2$.
- ii) easy calculations show that the sequence (P_n) is not monotone in n .

In this paper we shall generalize Van Zuijlen's result and derive sharp lower bound for probabilities concerning ξ standard deviations:

$$P_n := \mathbb{P}\{|S_n| \leq \xi\sqrt{n}\} = \mathbb{P}\{|\sum_{i=1}^n \epsilon_i| \leq \xi\sqrt{n}\}, \tag{2}$$

where $\xi \in (0, 1]$. Note that trivially

$$P_1 = \begin{cases} 1, & \text{for } \xi = 1; \\ 0, & \text{for } \xi < 1. \end{cases}$$

Throughout the paper n and k will denote nonnegative integers. Our result is as follows.

Theorem 1. Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be independent Rademacher random variables, so that

$$\mathbb{P}\{\epsilon_1 = 1\} = \mathbb{P}\{\epsilon_1 = -1\} = 1/2$$

and let S_n and P_n be defined as in (1) and (2), where $\xi \in (0, 1]$. Define

$$n_k = 2 \left\lceil \frac{\frac{k^2}{\xi^2} + k}{2} \right\rceil - k - 1, \quad C_k = \{n : n_k \leq n < n_{k+1}\}, \text{ and } Q_k^- := P_{n_{k+1}-1}.$$

Then, with Φ indicating the standard normal distribution function, we have for $k \geq 0$

- a. $P_n = \mathbb{P}\{|S_n| \leq \xi\sqrt{n}\} = \mathbb{P}\{|S_n| \leq k\}$, for $n \in C_k$,
- b. $Q_k^- = \min_{n \in C_k} P_n$,
- c. the sequence (Q_k^-) is strictly monotone increasing in k ,

Moreover,

- d. $\lim_{k \rightarrow \infty} Q_k^- = 1 - 2\Phi(\xi)$,
- e. $Q_1^- = P_{n_2-1} \leq P_n$, for all $n \geq n_1$.

A consequence of Theorem 1 is the following result.

Corollary 2. For $n \geq 2$ we have

$$P_n \geq \begin{cases} 1/2, & \text{for } \xi = 1; \\ 3/8, & \text{for } \xi \in [\sqrt{2/3}, 1); \\ 5/16, & \text{for } \xi \in [\sqrt{1/2}, \sqrt{2/3}). \end{cases}$$

More generally, if $0 < \xi \leq 1$ and $n_2 \geq 3$, n_2 odd, then we have

$$\frac{2}{\sqrt{n_2+1}} \leq \xi < \frac{2}{\sqrt{n_2-1}}, \quad n_1 = 2 \left\lceil \frac{n_2-3}{8} \right\rceil$$

and for all $n \geq n_1$

$$P_n \geq P_{n_2-1} = \binom{n_2-1}{(n_2-1)/2} 2^{-(n_2-1)}.$$

It is worthwhile to clarify in a plot the structure of the probabilities $P_n(\ell) = \mathbb{P}\{|S_n| = \ell\}$, where n and ℓ are nonnegative integers such that $n + \ell$ is even. See Figure 1.

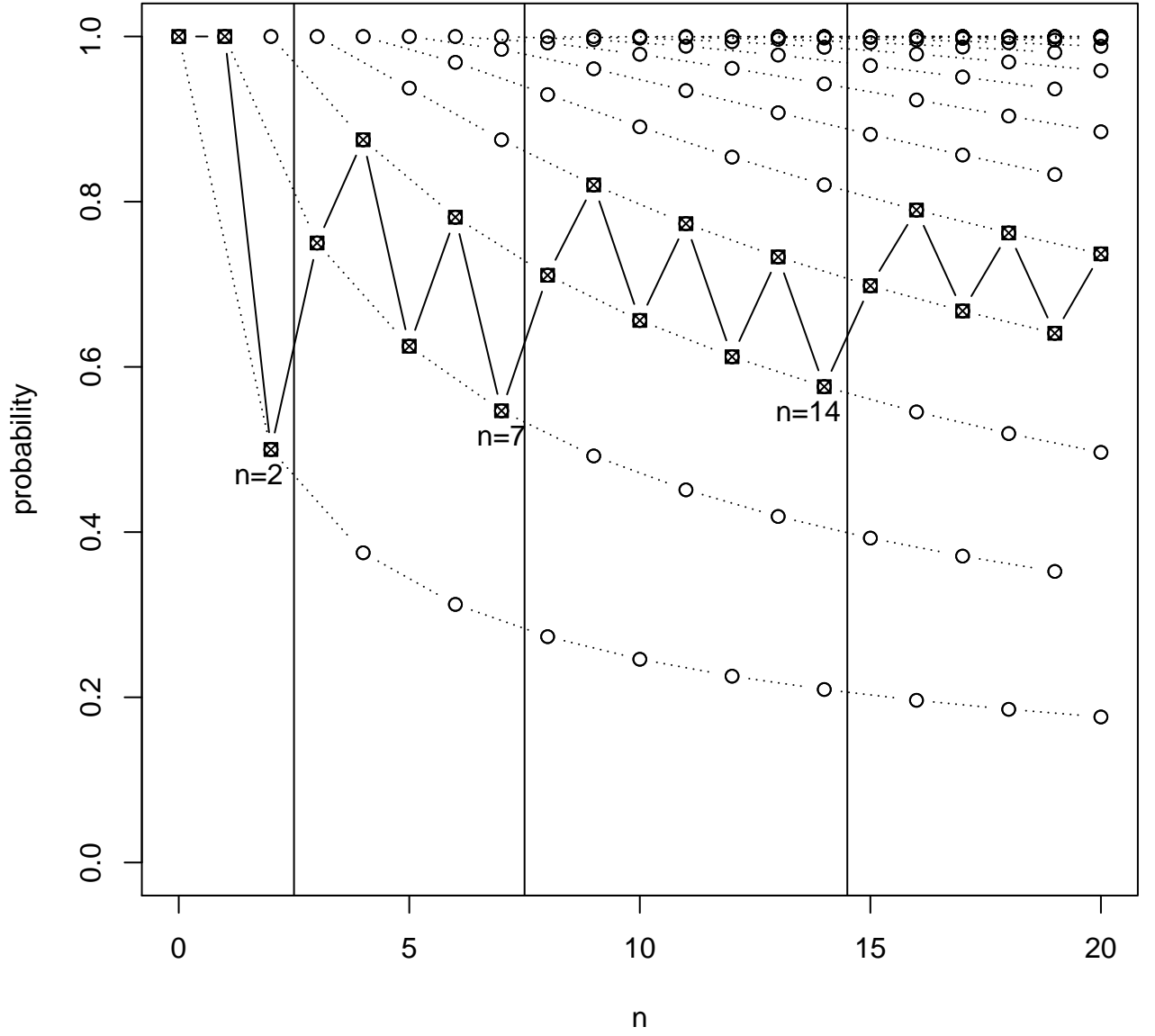


Figure 1: Graph of probabilities $P_n(\ell)$, $n+\ell$ even. (Dotted lines connect points with constant $\ell = 0, 1, 2, \dots$, upwards in graph. The square symbols indicate the points (n, P_n) for $\xi = 1$. The vertical lines separate the regions C_k , $k = 1, 2, \dots$)

2 Preliminaries

Be given independent Rademacher random variables ε_i , $i = 1, 2, 3, \dots$, as defined in Theorem 1, and let $S_n = \sum_{i=1}^n \varepsilon_i$ such that $S_0 = 0$. Define

$$P_n(k) = \mathbb{P}(|S_n| \leq k).$$

Since $n + S_n$ is even it follows that $P_n(k) = P_n(k-1)$ if $n+k$ is odd.

A basic property is the symmetry of the distribution of S_n :

$$\mathbb{P}\{S_n = k\} = \mathbb{P}\{S_n = -k\}.$$

Moreover, ε_n being independent of S_{n-1} ,

$$\begin{aligned} \mathbb{P}\{(S_{n-1} = k+1 \& \varepsilon_n = -1) \text{ or } (S_{n-1} = -k-1 \& \varepsilon_n = +1)\} \\ = 2\mathbb{P}\{S_{n-1} = k+1 \& \varepsilon_n = -1\} = \mathbb{P}\{S_{n-1} = k+1\} \end{aligned}$$

and, replacing ε_n by the equally distributed $-\varepsilon_n$,

$$\mathbb{P}\{(S_{n-1} = k+1 \& \varepsilon_n = +1) \text{ or } (S_{n-1} = -k-1 \& \varepsilon_n = -1)\} = \mathbb{P}\{S_{n-1} = k+1\}.$$

This leads to the following properties for $P_n(k)$.

Remark 3. Suppose $n+k$ is even, $n \geq 1$, then

$$\begin{aligned} P_n(k) &= P_{n-1}(k-1) + \mathbb{P}\{(S_{n-1} = k+1 \& \varepsilon_n = -1) \text{ or } (S_{n-1} = -k-1 \& \varepsilon_n = +1)\} \\ &= P_{n-1}(k-1) + \mathbb{P}\{S_{n-1} = k+1\}, \\ P_n(k) &= P_{n-1}(k+1) - \mathbb{P}\{(S_{n-1} = k+1 \& \varepsilon_n = +1) \text{ or } (S_{n-1} = -k-1 \& \varepsilon_n = -1)\} \\ &= P_{n-1}(k+1) - \mathbb{P}\{S_{n-1} = k+1\}. \end{aligned}$$

Suppose $n+k$ is even, $n \geq 1$, then

$$\begin{aligned} \mathbb{P}\{S_{n-1} = k-1\} &= \binom{n-1}{\frac{n+k}{2}-1} 2^{-(n-1)} = \frac{n+k}{n} \binom{n}{\frac{n+k}{2}} 2^{-n} = \frac{n+k}{n} \mathbb{P}\{S_n = k\}, \\ \mathbb{P}\{S_{n-1} = k+1\} &= \mathbb{P}\{S_{n-1} = -k-1\} = \frac{n-k}{n} \mathbb{P}\{S_n = -k\} = \frac{n-k}{n} \mathbb{P}\{S_n = k\}. \end{aligned}$$

Suppose $n+k$ is even, $n \geq 1$, $k-1 \geq 0$. It follows that

$$\begin{aligned} P_{n-1}(k-1) - P_{n+1}(k-1) &= P_n(k-2) + \mathbb{P}\{S_{n-1} = k-1\} - P_n(k-2) - \mathbb{P}\{S_n = k\} \\ &= \mathbb{P}\{S_{n-1} = k-1\} - \mathbb{P}\{S_n = k\} = \frac{k}{n} \mathbb{P}\{S_n = k\}. \end{aligned}$$

Furthermore, for $n \geq k \geq 0$,

$$\begin{aligned} \frac{\mathbb{P}\{S_n = k\}}{\mathbb{P}\{S_{n+2} = k\}} &= \frac{\mathbb{P}\{S_n = k\}}{\mathbb{P}\{S_{n+1} = k+1\}} \times \frac{\mathbb{P}\{S_{n+1} = k+1\}}{\mathbb{P}\{S_{n+2} = k\}} = \frac{n+1+k+1}{n+1} \times \frac{n+2-k}{n+2} \\ &= \frac{(n+2)^2 - k^2}{(n+2)^2 - (n+2)}. \end{aligned}$$

In particular, if $k^2 \leq n+2$ then $\mathbb{P}\{S_n = k\} \geq \mathbb{P}\{S_{n+2} = k\}$, with equality only if $k^2 = n+2$.

Corollary 4. Suppose $n + k$ is even, $n > k$ (i.e. $n \geq k + 2$), then $P_n(k) = P_n(k + 1) < P_m(k + 1)$ for all $m < n$.

Proof. According to the above Remark 3

$$P_n(k) < P_{n-2}(k) < \cdots < P_{k+2}(k) < P_k(k) = 1,$$

$$P_n(k) < P_{n-1}(k + 1) < P_{n-3}(k + 1) < \cdots < P_{k+1}(k + 1) = 1.$$

□

Theorem 5. Suppose $k \geq 1$, $n \geq k$ and $n + k$ is even. If $n + 2 \geq k^2$ and $\ell \geq 0$, such that $n + 1 + 2\ell < \frac{(k+1)^2}{k^2}(n + 2)$, then $P_n(k - 2) < P_{n+1+2\ell}(k - 1)$.

Proof. For $\ell = 0$ we remark that $P_{n+1}(k - 1) - P_n(k - 2) = \mathbb{P}\{S_n = k\} > 0$. For $\ell \geq 1$ it is sufficient to show $P_{n+1}(k - 1) - P_n(k - 2) > P_{n+1+2\ell}(k - 1) - P_{n+1+2\ell}(k - 1)$. Since

$$P_{n+1}(k - 1) - P_n(k - 2) = \mathbb{P}\{S_n = k\} \quad \text{and}$$

$$P_{n+1}(k - 1) - P_{n+1+2\ell}(k - 1) = \sum_{i=1}^{\ell} \frac{k}{n + 2i} \mathbb{P}\{S_{n+2i} = k\},$$

this inequality will follow from the claim $\mathbb{P}\{S_n = k\} > \sum_{i=1}^{\ell} \frac{k}{n+2i} \mathbb{P}\{S_{n+2i} = k\}$.

If $\ell = 1$, then $\mathbb{P}\{S_n = k\} \geq \mathbb{P}\{S_{n+2} = k\} > 0$ and $1 > \frac{k}{k+2} \geq \frac{k}{n+2}$. If $\ell > 1$ or $n + 2 > k^2$, then $\mathbb{P}\{S_n = k\} > \mathbb{P}\{S_{n+2\ell} = k\} > 0$, so that it is sufficient to show that $\sum_{i=1}^{\ell} \frac{k}{n+2i} \leq 1$. Theorem 5 now follows from Lemma 8 in the Appendix. □

Corollary 6. Let $n_k, k = 1, 2, 3, \dots$, be an increasing sequence of integers such that $n_1 \geq 0$, $n_k + k$ is odd, $n_k + 1 \geq k^2$ and $n_{k+1} - 1 < \frac{(k+1)^2}{k^2}(n_k + 1)$. Then for $n_k \leq m < n_{k+1} - 1$ we have

$$\mathbb{P}\{S_{n_2-1} = 0\} = P_{n_2-1}(0) < \cdots < P_{n_k-1}(k - 2) < P_{n_{k+1}-1}(k - 1) < P_m(k),$$

which for $k = 1$ reduces to

$$\mathbb{P}\{S_{n_2-1} = 0\} = P_{n_2-1}(0) < P_m(1).$$

Proof. For $k \geq 2$ or $n_1 \geq 2$, apply Theorem 5 with $n = n_k - 1$ and $\ell = (n_{k+1} - n_k - 1)/2$. In case $k = 1$ and $n_1 = 0$ we have $n_3 = 3$ and the claim in the corollary is trivial. □

3 The original context

Let $\xi > 0$ and consider the event $\{|S_n| \leq \xi\sqrt{n}\}$. Let k be the integer such that $n + k$ is even and $k \leq \xi\sqrt{n} < k + 2$. Then $\{|S_n| \leq \xi\sqrt{n}\} = \{|S_n| \leq k\}$. Notice that such k satisfies the inequalities

$$\frac{n + k}{2} \leq \frac{n + \xi\sqrt{n}}{2} < \frac{n + k + 2}{2} = \frac{n + k}{2} + 1$$

so that $\frac{n+k}{2} = \left\lfloor \frac{n+\xi\sqrt{n}}{2} \right\rfloor$ and hence

$$k = \kappa(n) := 2 \left\lfloor \frac{n + \xi\sqrt{n}}{2} \right\rfloor - n.$$

It follows immediately that $\kappa(n+2) \geq \kappa(n)$, $\kappa(0) = 0$. Moreover

$$\begin{aligned} \kappa(n+1) - \kappa(n) &= 2 \left\lfloor \frac{n+1 + \xi\sqrt{n+1}}{2} \right\rfloor - n - 1 - 2 \left\lfloor \frac{n + \xi\sqrt{n}}{2} \right\rfloor + n \\ &= 2 \left\lfloor \frac{n+1 + \xi\sqrt{n+1}}{2} \right\rfloor - 2 \left\lfloor \frac{n + \xi\sqrt{n}}{2} \right\rfloor - 1, \end{aligned}$$

so that $\kappa(n+1) - \kappa(n)$ is odd and greater than or equal to -1 .

It is interesting to notice the following fact: If a and b are nonnegative integers we have $\xi\sqrt{a} < \kappa(a) + 2$ and $\kappa(b) \leq \xi\sqrt{b}$, so that

$$a < \frac{(\kappa(a) + 2)^2}{\kappa(b)^2} b. \quad (3)$$

From the inequality $\lfloor a \rfloor - \lfloor b \rfloor < a - b + 1$ one concludes

$$\begin{aligned} \kappa(n+1) - \kappa(n) &< n+1 + \xi\sqrt{n+1} - n - \xi\sqrt{n} + 1 \\ &= 2 + \xi\sqrt{n+1} - \xi\sqrt{n} = 2 + \frac{\xi}{\sqrt{n+1} + \sqrt{n}}. \end{aligned}$$

It follows that for $\xi \leq 1$, $\kappa(n+1) - \kappa(n) \leq 1$, since then it is an odd number strictly less than 3. As a matter of fact, already for $\xi < 2\sqrt{2}$ we have $\kappa(n+1) - \kappa(n) \leq 1$. In the sequel assume that $\xi \leq 1$. Then we have the basic properties

$$\kappa(n+1) - \kappa(n) = \pm 1, \quad \kappa(0) = 0, \quad \kappa(n) \leq \xi\sqrt{n}, \quad \kappa(n+2) \geq \kappa(n).$$

For $k \geq 1$, define

$$n_k := \min\{n \mid \kappa(n+1) \geq k\} = 2 \left\lceil \frac{\frac{k^2}{\xi^2} + k}{2} \right\rceil - k - 1. \quad (4)$$

It is clear that n_k is strictly increasing in k . Moreover $\kappa(n_k + 1) = k$ and $\kappa(n_k) = k - 1$ and $\kappa(n_k - 1) = k - 2$ (if $k \geq 2$ or if $k = 1$ and $n_1 \geq 1$). Also $n_k + k$ is odd. In case $\xi = 1$ it is easy to see that $n_k = k^2 - 1$. Since n_k is decreasing in ξ , it follows for $\xi \leq 1$ that $n_k \geq k^2 - 1$.

Notice that for $m \leq n_{k+1}$ we have $\kappa(m) < k + 1$, so that $\kappa(m) \leq k$. On the other hand, if $\kappa(m) \leq k - 2$, it follows for all $n \leq m$ that $\kappa(n) \leq k - 1$, so that $m < n_k + 1$ and

since $\kappa(n_k) = k - 1$ it follows that $m < n_k$. We conclude that for $n_k \leq m \leq n_{k+1}$ we have $k - 1 \leq \kappa(m) \leq k$, so that

$$P_m(k) = \mathbb{P}\{|S_m| \leq \xi\sqrt{m}\}. \quad (5)$$

Since $k \leq \xi\sqrt{n_k + 1}$ and $0 < \xi \leq 1$ we have $k^2 \leq n_k + 1$. From Inequality (3) we obtain

$$n_{k+1} - 1 < \frac{(k+1)^2}{k^2}(n_k + 1).$$

Provided that $n_k - 1 \geq k$, Theorem 5 leads to the inequality $P_{n_k-1}(k-2) < P_{n_{k+1}-1}(k-1)$. Notice that $n_k \geq 2$ if $k \geq 2$ or if $k = 1$ and $\xi < 1$. The main result, Theorem 1, in fact follows from Corollary 6. More specifically,

Corollary 7. *Let $\xi \leq 1$ and $n_k, k = 1, 2, 3, \dots$ be defined as (4). Then, for $k \geq 2$ and all m satisfying $n_{k-1} \leq m < n_k$, we have*

$$\mathbb{P}\{S_{n_2-1} = 0\} = P_{n_2-1}(0) \leq P_{n_k-1}(k-2) < P_m(k-1) = \mathbb{P}\{|S_m| \leq \xi\sqrt{m}\}.$$

In particular, for $m \geq n_1$ we have $\mathbb{P}\{S_{n_2-1} = 0\} \leq \mathbb{P}\{|S_m| \leq \xi\sqrt{m}\}$, with equality only for $m = n_2 - 1$.

Proof of Theorem 1. Claim a) has been dealt with in (5). Claims b), c) and e) follow directly from the above Corollary 7. Finally, Claim d) follows from the Central Limit Theorem. \square

It is the condition $\xi \leq 1$ that implies that $n_k + 1 \geq k^2$, needed in Corollary 6. For $\xi > 1$ it is no longer true that $P_{n_{k+1}-1}(k-1) > P_{n_k-1}(k-2)$ as can be seen from the following examples. For $\xi = \sqrt{2}$, we have $n_4 = 7, n_5 = 12$ and $P_{n_5-1}(3) = \frac{99}{128} < \frac{100}{128} = P_{n_4-1}(2)$. For $\xi = 1.1$ and $k = 22$: $n_{22} = 399 = 20^2 - 1$; $n_{23} = 438$ and $P_{n_{23}-1}(21) < 0.70745 < P_{n_{22}-1}(20)$. For $\xi = 1.01$ and $k = 202$, $n_k = 39999 = 200^2 - 1$, $n_{k+1} = 40398$ and $P_{n_{203}-1}(201) < 0.6851152 < P_{n_{202}-1}(200)$.

Concerning Corollary 2 we note the following. It is straightforward to see that

$$n_2 = \begin{cases} 3, & \text{for } \xi = 1, \\ 5, & \text{for } \xi \in [\sqrt{2/3}, 1), \\ 7, & \text{for } \xi \in [\sqrt{1/2}, \sqrt{2/3}), \end{cases}$$

so that

$$\mathbb{P}\{-1 \leq S_{n_2-1} \leq 1\} = \begin{cases} 1/2, & \text{for } \xi = 1, \\ 3/8, & \text{for } \xi \in [\sqrt{2/3}, 1), \\ 5/16, & \text{for } \xi \in [\sqrt{1/2}, \sqrt{2/3}). \end{cases}$$

More generally, from definition (4) we have $n_2 = 2 \left\lceil \frac{\frac{4}{\xi^2} + 2}{2} \right\rceil - 2 - 1 = 2 \left\lceil \frac{2}{\xi^2} \right\rceil - 1$, which is equivalent to

$$\frac{2}{\sqrt{n_2 + 1}} \leq \xi < \frac{2}{\sqrt{n_2 - 1}}$$

and to

$$\frac{n_2 + 3}{8} < \frac{\frac{1}{\xi^2} + 1}{2} \leq \frac{n_2 + 5}{8}.$$

Since n_2 is odd, the open interval $(\frac{n_2+3}{8}, \frac{n_2+5}{8})$ does not contain an integer. Thus, for such ξ ,

$$n_1 = 2 \left\lceil \frac{\frac{1}{\xi^2} + 1}{2} \right\rceil - 1 - 1 = 2 \left\lceil \frac{n_2 + 5}{8} \right\rceil - 2 = 2 \left\lceil \frac{n_2 - 3}{8} \right\rceil$$

and for all $n \geq n_1$, we have from Theorem 1

$$P_n \geq P_{n_2-1} = \mathbb{P}\{S_{n_2-1} = 0\} = \binom{n_2 - 1}{(n_2 - 1)/2} 2^{-(n_2-1)}.$$

4 Examples

In case $\xi = \sqrt{1/2}$, we obtain for $k \in \{1, 2, \dots\}$

$$n_k = 2 \left\lceil \frac{\frac{k^2}{\xi^2} + k}{2} \right\rceil - k - 1 = 2 \left\lceil \frac{2k^2 + k}{2} \right\rceil - k - 1 = \begin{cases} 2k^2 - 1, & \text{for } k = \text{even}, \\ 2k^2, & \text{for } k = \text{odd}. \end{cases}$$

In this case $n_1 = 2, n_2 = 7, n_3 = 18, n_4 = 31$, so that $C_1 = [2, 6], C_2 = [7, 17], C_3 = [18, 30]$ and the minimal value in C_1 is

$$\begin{aligned} P_{n_2-1} &= P_6 = P\{-\xi\sqrt{n_2-1} \leq S_{n_2-1} \leq \xi\sqrt{n_2-1}\} = P\{-1 \leq S_6 \leq 1\} = P\{S_6 = 0\} = \\ &= P\{B_6 = 3\} = \frac{5}{16}. \end{aligned}$$

Here the B_n denote the binomial random variables as in the Introduction. Also,

$$P_{n_3-1} = P_{17} = \mathbb{P}\{-2 \leq S_{17} \leq 2\} = 2\mathbb{P}\{S_{17} = 1\} = 2\mathbb{P}\{B_{17} = 9\} = \frac{12155}{32768} \geq \frac{10240}{32768} = \frac{5}{16}.$$

In case $\xi = \sqrt{2/3}$, hence we obtain for $k \in \{1, 2, \dots\}$

$$n_k = 2 \left\lceil \frac{\frac{3k^2}{2} + k}{2} \right\rceil - k - 1 = 2 \left\lceil \frac{3k^2 + 2k}{4} \right\rceil - k - 1 = \begin{cases} \frac{3}{2}k^2 - 1, & \text{for } k = \text{even}, \\ \frac{3}{2}k^2 + \frac{1}{2}, & \text{for } k = \text{odd}. \end{cases}.$$

Hence, $n_1 = 2, n_2 = 5, n_3 = 14, n_4 = 23, n_5 = 38, n_6 = 53$ with blocks $C_1 = [2, 4], C_2 = [5, 13], C_3 = [14, 22], C_4 = [23, 37], C_5 = [38, 52]$. The minimal value in C_1 is obtained for

$$\begin{aligned} P_{n_2-1} &= P\{-\xi\sqrt{n_2-1} \leq S_{n_2-1} \leq \xi\sqrt{n_2-1}\} = P\{-1 \leq S_4 \leq 1\} = P\{S_4 = 0\} = \\ &= P\{B_4 = 2\} = \frac{3}{8}. \end{aligned}$$

Also,

$$P_{n_3-1} = \mathbb{P}\{-2 \leq S_{13} \leq 2\} = 2\mathbb{P}\{S_{13} = 1\} = 2\mathbb{P}\{B_{13} = 7\} = \frac{429}{1024} \geq \frac{384}{1024} = \frac{3}{8}.$$

In case $\xi = 1$ we obtain for $k \in \{1, 2, \dots\}$

$$n_k = 2 \left\lceil \frac{\frac{k^2}{\xi^2} + k}{2} \right\rceil - k - 1 = k^2 - 1.$$

We obtain for integers $k \geq 2$, $C_k = \{k^2 - 1, k^2, \dots, (k+1)^2 - 2\}$, with length $m_k = 2k + 1$. Now $n_1 = 0, n_2 = 3, n_3 = 8, n_4 = 15$, so that $C_1 = [0, 2], C_2 = [3, 7], C_3 = [8, 14]$. The minimal value in C_1 is obtained for

$$\begin{aligned} P_{n_2-1} &= P\{-\xi\sqrt{n_2-1} \leq S_{n_2-1} \leq \xi\sqrt{n_2-1}\} = P\{-1 \leq S_2 \leq 1\} = P\{S_2 = 0\} = \\ &= P\{B_2 = 1\} = \frac{1}{2}. \end{aligned}$$

The minimal value in C_2 is obtained for $n = n_3 - 1 = 7$ and equals

$$P_{n_3-1} = P_7 = \mathbb{P}\{-2 \leq S_7 \leq 2\} = 2\mathbb{P}\{S_7 = 1\} = 2\mathbb{P}\{B_7 = 4\} = \frac{35}{64} \geq \frac{32}{64} = \frac{1}{2}.$$

Appendix

In this section we state and prove the lemma needed in the proof of Theorem 5.

Lemma 8. *Suppose $k \geq 1$ and $n + k$ even. If $n + 2 \geq k^2$ and $\ell \geq 0$, such that $n + 1 + 2\ell < \frac{(k+1)^2}{k^2}(n + 2)$, then $\sum_{i=1}^{\ell} \frac{k}{n+2i} \leq 1$.*

Proof. The goal is to prove the inequality

$$\sum_{i=1}^{\ell} \frac{k}{n+2i} \leq 1.$$

It is easy to see that $k/(n+2i) + k/(n+2\ell+2-2i)$ is decreasing in i for $i \leq \ell/2$. Therefore it is sufficient to prove $k/(n+2) + k/(n+2\ell) < 2/\ell$, or equivalently

$$\frac{\ell}{n+2} + \frac{\ell}{n+2\ell} \leq \frac{2}{k}. \quad (6)$$

Since the left hand is increasing in ℓ , it is sufficient for given n to consider the maximally allowed ℓ . In the same way, given ℓ it is sufficient to prove the inequality for the minimally allowed n .

The condition $n + 1 + 2\ell < \frac{(k+1)^2}{k^2}(n + 2)$ is equivalent to $2\ell - 1 < \frac{2k+1}{k^2}(n + 2)$. Thus for any n such that $n + 2 \geq k^2$, $\ell = k$ is an allowed value for ℓ . The corresponding minimal value of n is $n = k^2 - 2$. It follows that for $\ell \leq k$ Inequality (6) holds:

$$\frac{2}{k} - \frac{\ell}{n+2} - \frac{\ell}{n+2\ell} \geq \frac{2}{k} - \frac{k}{n+2} - \frac{k}{n+2k} \geq \frac{2}{k} - \frac{k}{k^2} - \frac{k}{k^2} = 0.$$

Next consider the case $\ell \geq k + 1$. Then the condition $2\ell - 1 < \frac{2k+1}{k^2}(n + 2)$ leads to

$$n + 2 > \frac{2\ell - 1}{2k + 1}k^2 = k^2 + (\ell - k - 1)(k - \frac{1}{2}) + \frac{\ell - k - 1}{2(2k + 1)} \geq k^2 + (\ell - k - 1)(k - \frac{1}{2}). \quad (7)$$

In case $\ell = k + 1$ it means that $n + 2 > k^2$, and because $n + k$ is even, $n + 2 \geq k^2 + 2$. Substituting $\ell = k + 1$ and $n = k^2$ we get Inequality (6) for $\ell = k + 1$:

$$\frac{2}{k} - \frac{\ell}{n+2} - \frac{\ell}{n+2\ell} = \frac{2(k^2 + 2k + 4)}{k(n+2)(n+2\ell)} \geq 0.$$

For the case $\ell \geq k + 2$ we conclude from Inequality (7) that $n + 2 \geq k^2 + (\ell - k - 1)(k - \frac{1}{2}) + \frac{1}{2}$. Substituting $\ell = k + 2 + j$ and $n = k^2 + (\ell - k - 1)(k - \frac{1}{2}) - \frac{3}{2}$ we get

$$\frac{2}{k} - \frac{\ell}{n+2} - \frac{\ell}{n+2\ell} = \frac{2j(k^2 - 2) + j^2(2k - 3)}{k(n+2)(n+2\ell)}.$$

Since the right hand side is nonnegative for $j \geq 0$ and $k \geq 2$ we established Inequality (6) for $k \geq 2$ and $\ell \geq k + 2$.

If $k = 1$, n odd, then from $n + 1 + 2\ell < \frac{(k+1)^2}{k^2}(n + 2)$ it follows that $2\ell - 1 < 3(n + 2)$, which implies $2\ell - 1 \leq 3(n + 2) - 2$, so that the maximal ℓ is $\ell = (3n + 5)/2$. Again

$$\frac{2}{k} - \frac{\ell}{n+2} - \frac{\ell}{n+2\ell} = \frac{(n+1)(n+5)}{2(n+2)(n+2\ell)} \geq 0.$$

□

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